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Article

CreditRisk+ Model with Heterogeneous Obligors

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Abstract: The purpose of the paper, the main research process and the methods adopted, the main results and important conclusions should be expressed clearly in concise and clear language. If possible, mention as stated in Basel accord, the Creditrisk+ is a main tool for running stress testing in a credit portfolio. There are some modifications to original format of this protocol. In the current manuscript, in a sub-credit portfolio with heterogeneous obligors, with the same nonrandom probability of default, some limiting behaviors of total loss of portfolio are studied. In the case of random and correlated probabilities of default with ARTA models (autoregressive to any things model), by the Eigen analysis, some other limiting distributions are proposed. Finally, simulation results are proposed to verify some parts of the limiting results.

Keywords: ARTA Models; Basel Accord; Creditrisk+; Eigen Analysis; Heterogeneous Obligors; Limiting Distributions; Probability of Default; Total Loss

1. Introduction

Traditionally, there are three main approaches in the literature to quantify the credit risk of a credit portfolio. They are (i) financial engineering-based distance to default of Merton, (ii) the use of financial ratios and analyzing them using econometrics methods such as panel and logistic regressions, and (iii) the actuarial method referred to Creditrisk+. These models help financial institutions such as banks to evaluate their risk profiles and manage them by computing risk measures such as value at risk and running credit stress test according to the Basel accord guidelines. For a comprehensive review in Creditrisk+ and its extensions, see Gundlach and Lehrbass (2020).

In this paper, some modifications are proposed to distribution of total loss of credit portfolio. Creditrisk+ model considers the sum

$$L_n = \sum_{j=1}^{n} v_{jn} I_{jn}$$

as the normalized total loss of a given credit portfolio where there are n obligors at which $j-$th obligor defaults $I_{jn} = 1$ or zero, otherwise, see [1]. The normalized exposure of $j-$th obligor is $v_{jn} \in (0,1)$. It is a non-random value. Suppose that the probability of default for all obligors is p_n. This assumption happens in practice in a sub-portfolio with heterogeneous obligors in probability of default obtained by data mining clustering technique such as k-means, see [2]. It is also assumed that

$$p_n \to 0 \text{ as } n \to \infty,$$

see [3]. In this note, motivated by [4], the limiting behaviors of L_n for large n's are studied as follows:

a) Non-random exposures.

Here, it is assumed that exposures are non random variables. Two cases are proposed:

a_1) Limiting case. Let

$$v_{jn} = v_n(\frac{j}{n})$$

for some real values function $v_n(t)$ which converges to some $v(t)$ uniformly. In practice, this is done using spline technique of numerical analysis or basis function representation in topological data analysis; see [5]. Let

$$X_n(t) = \frac{n^{-\frac{1}{2}}}{\sqrt{p_n(1-p_n)}}\sum_{j=1}^{[nt]}(I_{jn} - p_n),$$
$$t \in (0,1).$$

Here, $X_n(t)$ converges in distribution to standard Brownian motion on topological space $D[0,1]$, see Billingsley (1968). Notation [.] stands for the integer-value function. Notice that

$$L_n = \sum_{j=1}^{n} v_{jn}(I_{jn} - p_n) + p_n \sum_{j=1}^{n} v_{jn} =$$
$$= \sqrt{np_n(1-p_n)} \times$$
$$\times \sum_{j=1}^{n} v_n\left(\frac{j}{n}\right)\{X_n\left(\frac{j}{n}\right) - X_n\left(\frac{j-1}{n}\right)\} + np_n\frac{\sum_{j=1}^{n} v_{jn}}{n} =$$
$$= \sqrt{np_n(1-p_n)} \int_0^1 v_n(t)\, dX_n(t) + np_n \int_0^1 v_n(t)dt.$$

Let $n \to \infty$, assuming

$$np_n \to \lambda > 0$$

and $1 - p_n \to 0$, then L_n converges in distribution to

$$\sqrt{\lambda}\int_0^1 v(t)\, dW(t) + \lambda\int_0^1 v(t)dt =$$
$$\sqrt{\lambda}\int_0^1 v(t)\, dB(t),$$

where

$$B(t) = W(t) + \sqrt{\lambda}t$$

is the Brownian motion with a drift. As $n \to \infty$, then L_n has normal distribution with mean $\lambda\int_0^1 v(t)\, dt$ and variance $\lambda\int_0^1 v^2(t)\, dt$. Thus,

$$P(L_n \le l) \to \Phi\left(\frac{l-\lambda\int_0^1 v(t)dt}{\sqrt{\lambda\int_0^1 v^2(t)dt}}\right).$$

a_2) Optimal control. Suppose that $dv = a(t, v, u)dt$ represents a differential equation for $v(t)$. To maximize $P(L_n \le l)$ with respect to $v(t, u)$, it is enough to minimize

$$\int_0^1 v^j(t, u)\, dt, j = 1,2.$$

Let τ be the importance weight of

$$\int_0^1 v(t, u)\, dt,$$

therefore, the optimal control problem is given by

$$\min_u \int_0^1 (v^2(t, u) + \tau v(t, u))dt,$$

with respect v assuming

$$dv = a(t, v, u)dt.$$

The rest of paper is organized as follows. In the next section, some theoretical expansions are proposed. Simulation results are given in section 3.

2. Extensions

In this section, some extensions to proposed problem in section 1 are given.

b) Random exposure case.

Here, it is assumed that exposures are random variables. Again, two cases are proposed:

b_1) Uniform exposures. Assume that $v_{jn}, j = 1, \dots, n$ are independent random variables uniformly distributed on (0,1). Let $v_{1:n}, \dots, v_{n:n}$ be the order statistic of $v_{jn}, j = 1, \dots, n$. Rewrite

$$L_n = \sum_{j=1}^{n} v_{j:n} I_{j:n}$$

where $I_{j:n}$ the indicator is corresponding to $v_{j:n}$. Let $E_j; j = 1, \dots, n+1$ be a sequence of $n+1$ independent exponentially distributed random variables. One can see that

$$(v_{1:n}, \dots, v_{n:n}) =^d \left(\frac{E_1}{\sum_{j=1}^{n+1} E_j}, \frac{E_1 + E_2}{\sum_{j=1}^{n+1} E_j}, \dots, \frac{\sum_{j=1}^{n} E_j}{\sum_{j=1}^{n+1} E_j}\right)$$

Therefore, S_n is distributed as

$$\sum_{j=1}^{n} \frac{\sum_{k=1}^{j} E_k}{\sum_{k=1}^{n+1} E_k} I_{j:n}.$$

Following the above scheme, it can be seen that L_n converges in distribution to

$$\sqrt{\lambda} \int_0^1 \Gamma(t)\, dB(t),$$

where $\Gamma(t)$ is the gamma process. For other distribution of v_{jn}, the beta distribution has flexible shape to fit to v_{jn}. Then, the Monte Carlo simulation and kernel density estimation are powerful methods to fit distribution to L_n.

b_2) Non-uniform exposures. Here, the uniform assumption of exposures is relaxed and instead distribution of an upper bound of L_n is approximated. Notice that

$$L_n \geq v_{1:n} N_n$$

where N_n has the limiting Poisson distribution with parameter λ. Indeed,

$$P(L_n \leq x) \leq P(v_{1:n} N_n \leq x).$$

As follows, an approximate distribution is derived for $v_{1:n}$. Let F be the distribution function of $v_{1:n}$ and g be the first natural number at which the g −th derivative of F computed at zero is not zero, i.e.,

$$F^{(g)}(0) \neq 0.$$

Consider

$$w_n = n^{1/g} v_{1:n}.$$

Let

$$\zeta_k = F^{(k)}(0)/k!,$$
$$k = g, g+1, \dots\ .$$

Note that the distribution function of w_n is

$$F_n(w) = 1 - \left(1 - F\left(\frac{w}{n^{1/g}}\right)\right)^n.$$

Using the Taylor series of $F\left(\frac{w}{n^{1/g}}\right)$ around zero, it is seen that

$$F\left(\frac{w}{n^{1/g}}\right) = \frac{\zeta_g w^g}{n} + \frac{\zeta_{g+1} w^{g+1}}{n^{1+\frac{1}{g}}} + \cdots\ .$$

Substituting this expression in $F_n(w)$ and keeping term $\frac{\zeta_g w^g}{n}$ and removing all the next terms in right side hand of above equation, it is seen that

$$F_n(w) = 1 - (1 - \frac{\zeta_g w^g}{n})^n.$$

Assuming $n \to \infty$, it concludes that $F_n(w)$ converges to

$$F(w) = 1 - e^{-\zeta_g w^g} =$$
$$= 1 - \exp\{-(\frac{w}{\zeta_g^{-1/g}})^g\}.$$

Thus,

$$w_n = n^{1/g} v_{1:n}$$

has the Weibull distribution with scale and shape parameters $\zeta_g{}^{-1/g}$, g, respectively.
Assuming

$$P(L_n \le x_\theta) = \theta,$$

therefore,

$$P(v_{1:n} N_n \le x_\theta) \ge \theta.$$

To find x_θ, samples N_n from Poisson distribution with parameter λ are generated. Again, samples w_n from Weibull distribution with parameters

$$\zeta_g{}^{-1/g}, \ g$$

are generated and

$$v_{1:n} = n^g w_n$$

are computed. Then, $v_{1:n}N_n$'s are computed. The first point at which the empirical distribution function of $v_{1:n}N_n$ is larger than θ is the x_θ.

c) Random p_n. Creditrisk+ extended model assumes that the probability of default is random variable, hence assumes a statistical structure for it. Assuming p_n is a random variable such that np_n converges in probability to λ, i.e.,

$$np_n \to^p \lambda.$$

One can see that

$$E(L_n|p_n) = \frac{\sum_{i=1}^n v_n\left(\frac{i}{n}\right)}{n} np_n \to^p \lambda \int_0^1 v(t)\, dt$$

$$var(L_n|p_n) =$$
$$= \frac{\sum_{i=1}^n v_n^2\left(\frac{i}{n}\right)}{n} np_n(1-p_n) \to^p$$
$$\to^p \lambda \int_0^1 v^2(t)\, dt.$$

Following [5], let $p_n = \pi_n \sum_{j=1}^M \omega_{nj}\gamma_j$ be a linear combination of M independent risk factors represented by M gamma distributed random variables γ_j with parameters β_j^{-1} and β_j, $1 \le j \le M$. Here, ω_{nj} is the share of j-th risk factor and π_n is the normalized factor. Rewrite p_n as

$$p_n = \sum_{j=1}^M \pi_n \omega_{nj} \beta_j \Gamma_j.$$

In the special case, suppose that

$$\omega_{nj} = \frac{\beta}{n\pi_n \beta_j}$$

for some $\beta > 0$ and Γ_j's are independent gamma distributed random variables with parameters β_j^{-1} and 1. Here,

$$np_n = \lambda = \beta \sum_{j=1}^M \Gamma_j$$

has gamma distribution with parameters $\sum_{j=1}^M \beta_j^{-1}$ and β^{-1}. This distribution plays the role of prior distribution. Beside this, notice that $L_n|\lambda$ has normal distribution with mean

$$\lambda \int_0^1 v(t)dt$$

and variance

$$\lambda \int_0^1 v^2(t)dt$$

and λ has gamma distribution with parameters $\sum_{j=1}^{M} \beta_j^{-1}$ and β^{-1}. This is the likelihood function after observing the loss L_n. Thus, using the Bayesian theorem, the posterior distribution is achievable. Here, the Monte Carlo Markov Chain (MCMC) method may be applied to propose the posterior estimate of λ (equivalently p_n).

[6] proposed a simple method for running MCMC method. Given observed loss $L_n = l$, samples are generated from posterior density $g(\lambda|l)$ using weighted bootstrap as follows:

1. Generate $\lambda_i, i = 1, \dots, B$ from prior gamma distribution.

2. Compute bootstrap weights

$$q_i = \frac{L(\lambda_i; l)}{\sum_{j=1}^{B} L(\lambda_j; l)}, i = 1, \dots, B$$

where $L(\lambda; l)$ is the likelihood (density) function of normal distribution with mean and variance

$$\lambda \int_0^1 v(t)\, dt,\ \lambda \int_0^1 v^2(t)\, dt,$$

respectively computed at l.

3. Generate bootstrapped samples λ_i^* using weights q_i's and compute the sample mean $\bar{\lambda}^*$.

4. Repeat steps 1-3 B times to derive $\bar{\lambda}^*{}_b; b = 1, \dots, B$ and finally compute

$$\bar{\lambda}^{*B} = \frac{\sum_{b=1}^{B} \bar{\lambda}^*{}_b}{B}.$$

The maximum a posteriori (MAP) estimates of λ is proposed by maximizing $g(\lambda)L(\lambda; l)$ where $g(\lambda)$ is the density of gamma distribution with parameters 1,1. It is equivalent to maximize

$$\lambda^{-1/2} \exp\left\{\frac{-1.5l^2}{\lambda} - \frac{11}{8}\lambda\right\}.$$

It is concluded that

$$\lambda = \frac{\sqrt{1 + 33l^2} - 1}{5.5}.$$

d) ARTA p_n's. Creditrisk+ extended model assumes that there is correlation between probability of default of obligors exist in the credit portfolio. Thus, by adding new comer obligors the value of p_n is changed, significantly. In the literature, a strong approach for modeling correlations in a portfolio is the copula function approach. However, in the current paper, motivated by [7], the autoregressive to anything (ARTA) model is applied.

Furthermore, we assume that as soon as new comer obligors enter to heterogeneous credit portfolio, the portfolio is still heterogeneous, however, the common probability of default is changed. Hereafter, two types of ARTA models are proposed.

d_1) ARTA: type 1. Consider

$$p_n = \pi_n \sum_{j=1}^{M} \omega_{nj}\gamma_j.$$

Let

$$\omega_n = (\omega_{n1}, \dots, \omega_{nM})'$$
$$\gamma = (\gamma_1, \dots, \gamma_M)'$$

and note that

$$p_n = \pi_n \omega_n' \gamma.$$

It is easy to see that

$$p_n = \frac{\pi_n}{\pi_{n-1}} p_{n-1} + \pi_n(\omega_n - \omega_{n-1})'\gamma.$$

To make sure that np_n converges in probability, it is enough to suppose that $n\pi_n$ goes to zero and $\omega_n - \omega_{n-1}$ goes to zero. As special case, assume that $\omega_{nj} = \omega_j$ is independent of n and let

$$p_n = \frac{1}{n}\sum_{j=1}^{M} \omega_j \gamma_j.$$

Generally, let

$$a_n = \frac{\pi_n}{\pi_{n-1}}$$

which converges to 1 and

$$b_n = \pi_n(\omega_n - \omega_{n-1})'$$

tends to zero. Therefore,

$$p_n = a_n p_{n-1} + b_n'\gamma.$$

This is a type of ARTA models. Let $\lambda_n = np_n$.
Notice that

$$\lambda_n = \frac{n}{n-1} a_n \lambda_{n-1} + nb_n'\gamma.$$

The variance of $nb_n'\gamma$ is also an important quantity to make sure that λ_n converges in probability and it is necessary that variance of $nb_n'\gamma$ is close to zero. Notice that

$$var(nb_n'\gamma) = n^2 b_n' var(\gamma) b_n.$$

As nb_n goes to zero, therefore, $var(nb_n'\gamma)$ gets small. Hereafter, the variance matrix $var(\gamma)$ and its empirical estimate are studied. Let $(\tau_i, e_i), i = 1, \dots, M$ be the eigenvalues and eigenvectors of $var(\gamma)$. Therefore,

$$var(\gamma) = \sum_{i=1}^{M} \tau_i e_i e_i'.$$

Therefore,

$$var(nb_n'\gamma) = \sum_{i=1}^{M} \tau_i (nb_n' e_i)^2.$$

Following principal component analysis (PCA), let $m \ll M$ such that

$$\frac{\sum_{i=1}^{m} \tau_i (nb_n' e_i)^2}{\sum_{i=1}^{M} \tau_i (nb_n' e_i)^2} \approx 0.95.$$

Therefore,

$$\sum_{i=1}^{m} \tau_i (nb_n' e_i)^2$$

is good approximation for empirical estimate of $var(nb_n'\gamma)$. [8] used the Eigen analysis of PCA to noise filtering in correlation matrix. Hence, quantity

$$\sum_{i=m+1}^{M} \tau_i (nb_n' e_i)^2$$

relates to noise of empirical estimate. This technique has two advantages i.e., the dimension reduction (since $m \ll M$) and variability reduction of empirical estimate of $var(nb_n'\gamma)$. [9] proposed two methods to approximate empirical estimate of $var(\gamma)$ by Toeplitz matrix of AR(1) process. They also suggest the Toeplitz approximation with discrete cosine transform for simplification of computations.

d_2) ARTA: type 2. Other type of ARTA model may be proposed. Let a_n is close to 1 and $\zeta_n = b_n'\gamma$, then

$$p_n = p_{n-1} + \zeta_n$$

where

$$\mathrm{var}(\zeta_n) \le \frac{1}{n^{2+\delta}}$$

for some $\delta > 0$. Let $\lambda_n = np_n$. Therefore,

$$\lambda_n - \lambda_{n-1} = p_{n-1} + \zeta_n.$$

Thus,

$$E|\lambda_n - \lambda_{n-1}|^2 \le \frac{1}{n^{\delta}} \to 0.$$

Therefore,

$$\lambda_n \to^p \lambda.$$

Let

$$x_n = \Phi^{-1}(p_n),$$

where Φ stands for the distribution function of standard normal distribution and Φ^{-1} is its inverse. Define

$$x_n = x_{n-1} + \varepsilon_n,$$

where ε_n has normal distribution with zero mean and variance less than $\frac{1}{n^{2+\delta}}$.

<u>e) AR(1) approximation</u>. In the case of correlated risk factors, the Credirisk+ assumes the risk factors γ_i given latent variable Γ are independent and each γ_i has gamma distribution with shape parameter Γ/β_i and scale parameter β_i. The Γ, itself, has gamma distribution with parameters σ^{-2} and σ^2. It is seen that the unconditional variance of γ_i is $\sigma^2 + \beta_i$ and unconditional covariance between γ_i and γ_j is σ^2. Thus, the unconditional correlation is

$$\rho_{ij} = \frac{\sigma^2}{(\sigma^2+\beta_i)^{0.5}(\sigma^2+\beta_j)^{0.5}} =$$
$$= (1+\beta_i/\sigma^2)^{-0.5}(1++\beta_j/\sigma^2)^{-0.5} =$$
$$= \exp\left(-\frac{1}{2}\left(\log\left(1+\frac{\beta_i}{\sigma^2}\right)+\log\left(1+\frac{\beta_j}{\sigma^2}\right)\right)\right).$$

Using approximation $\log(1+x) \approx x$, it is seen that

$$\rho_{ij} \approx \exp\left(-\frac{\beta_i+\beta_j}{2\sigma^2}\right).$$

Here, using the [10], the correlation matrix

$$\mathrm{cor}(\gamma) = (\rho_{ij})_{i,j=1}^{M}$$

is approximated by employing AR(1) model with Toeplitz correlation matrix which is given by

$$(a_{ij})_{i,j=1}^{M} = (a^{|i-j|})_{i,j=1}^{M}$$

for some $0 < a < 1$. Let

$$a = \exp\left(-\frac{\theta}{2\sigma^2}\right)$$

for some $\theta > 0$. Notice that

$$\frac{\rho_{ij}}{a_{ij}} = \exp\left(-\frac{(\beta_i+\beta_j)\theta - |i-j|}{2\sigma^2}\right).$$

To minimize

$$\left|\frac{\rho_{ij}}{a_{ij}} - 1\right|, \text{ for all } i,j,$$

it suffices to choose θ such that

$$\sum_{i=1}^{M}(\beta_i-\theta i)^2$$

is minimized. Indeed,

$$\theta=\frac{\sum_{i=1}^{M} i\beta_i}{\sum_{i=1}^{M} i^2}.$$

As soon as, θ converges to zero, or σ^2 tends to infinity, then Toeplitz correlation matrix is approximated by discrete cosine transformation matrix, see [11].

f) Adaptive filter for p_n. Here, motivated [12], a Kalman filter is derived for p_n. For large n's, the $\hat{p}_n$ has normal distribution with mean p_n and approximated variance

$$\frac{\hat{p}_n(1-\hat{p}_n)}{n}$$

(using Slutsky theorem). Notice that

$$p_n=a_n p_{n-1}+\sum_{j=1}^{M} b_{nj}\gamma_j,$$

where γ_j has gamma distribution with mean 1 and variance σ_j^2 and

$$b_{nj}=\pi_n(\omega_{nj}-\omega_{n-1,j})$$

at which

$$a_n=\frac{\pi_n}{\pi_{n-1}}.$$

Following [10], suppose that the posterior mean and variance of p_{n-1} are

$$\varphi_{n-1} \text{ and } \eta_{n-1}^2.$$

Therefore, the mean and variance of prior of p_n are

$$a_n\varphi_{n-1}+\kappa_{1n},\ a_n^2\eta_{n-1}^2+\kappa_{2n},$$

respectively, where

$$\kappa_{1n}=\sum_{j=1}^{M} b_{nj},$$
$$\kappa_{2n}=\sum_{j=1}^{M} b_{nj}^2.$$

Thus, the updating Kalman-type estimates are given by

$$\varphi_n=(1-z_n)a_n(\varphi_{n-1}-\kappa_{1n})+z_n\hat{p}_n,$$

where

$$z_n=\frac{a_n^2\eta_{n-1}^2+\kappa_{2n}}{a_n^2\eta_{n-1}^2+\kappa_{2n}+\frac{\hat{p}_n(1-\hat{p}_n)}{n}},$$

and

$$\frac{1}{\eta_n^2}=\frac{n}{\hat{p}_n(1-\hat{p}_n)}+\frac{1}{a_n^2\eta_{n-1}^2+\kappa_{2n}}.$$

In practice, distribution of

$$\sum_{j=1}^{M} b_{nj}\gamma_j$$

is approximated by a chi-squared distribution using the Satterthwaite method and quantities $\kappa_{in}, i=1,2$ are computable, more easily.

g) Number of defaults. Let

$$I_n=\sum_{i=1}^{n} J_i$$

be the numbers of defaults in a credit portfolio. The I_n/n is the empirical estimate of probability of default p_n. Here, the behavior of I_n is studied. In the case of random p_n's, given p_n, the I_n has binomial distribution with parameters n and p_n. Form the Bayesian inference point of view, some prior distributions for p_n are required. Creditrisk+ considers a normalized (with normalizing factor

π_n) linear combination of gamma distributed risk factors γ_i's with unit means and variance σ_i^2, that is

$$p_n = \pi_n \sum_{i=1}^{M} w_{ni}\gamma_i.$$

In special cases that $\pi_n w_{ni}$ is independent of n or using the Satterthwaite approximation (see [11]), it can be seen that p_n has the exact or approximated gamma distribution. Let $\mu_n = E(p_n)$ and $\vartheta_n^2 = var(p_n)$. Here, to simplify the computation of Bayesian posterior distribution of p_n, the conjugate beta prior distribution

$$beta(\theta_{1n}, \theta_{2n})$$

is chosen. Because of flexibility of shape of beta densities, see [11], using the matching moment method, θ_{1n}, θ_{2n} are selected such that the first two non-central moments of $beta(\theta_{1n}, \theta_{2n})$ and p_n are matched, i.e.,

$$\frac{\theta_{1n}}{\theta_{1n} + \theta_{2n}} = \mu_n$$

$$\frac{\theta_{1n}(\theta_{1n}+1)}{(\theta_{1n}+\theta_{2n})(\theta_{1n}+\theta_{2n}+1)} = \vartheta_n^2.$$

It is easy to see that

$$\theta_{1n} = c_n\mu_n$$
$$\theta_{2n} = c_n(1-\mu_n),$$

where

$$c_n = \frac{\mu_n(1-\mu_n)}{\vartheta_n^2} - 1.$$

In this way, the gamma density of p_n is well approximated by

$$beta(\theta_{1n}, \theta_{2n})$$

density. For posterior stage, motivated by [12], notice that using the Bayesian rule, the posterior distribution is beta distribution with parameters

$$\theta_{1n}^* = I_n + \theta_{1n}$$
$$\theta_{2n}^* = n - I_n + \theta_{2n}.$$

These parameters are updated using the following rules:

$$\theta_{1n}^* = \theta_{1,n-1}^* + J_n + \delta_{1n},$$
$$\theta_{2n}^* = \theta_{2,n-1}^* + (1 - J_n) + \delta_{2n},$$

where

$$\delta_{1n} = \theta_{1n} - \theta_{1,n-1}$$
$$\delta_{2n} = \theta_{2n} - \theta_{2,n-1}.$$

For prediction of I_{n+1}, using the Bayesian predictive posterior expectation, notice that

$$E(I_{n+1}|I_1, \dots, I_n) = \frac{\theta_{1n}^*}{\theta_{1n}^* + \theta_{2n}^*} =$$

$$= \zeta_n\left(\frac{I_n}{n}\right) + (1-\zeta_n)\frac{\theta_{1n}}{\theta_{1n} + \theta_{2n}},$$

which is a weighted average (mixture) of empirical estimate $\frac{I_n}{n}$ and prior estimate

$$\frac{\theta_{1n}}{\theta_{1n} + \theta_{2n}},$$

of p_n with mixing weight $\zeta_n = \frac{n}{n+\theta_{1n}+\theta_{2n}}$.

3. Simulations

Here, some parts of above theoretical results are simulated.

Case 1. Let v_{jn}'s are independent and uniformly distributed. To find function $v_n(t)$ such that

$$v_{jn} = v_n(j/n);$$

the regression method is applied. To this end, let

$$v_n^{-1}(v_{jn}) = \frac{j}{n} + \varepsilon_{jn}; j = 1, \dots, n.$$

For $n = 100$, the scatter plot of

$$\sum_{j=1}^{[nt]} \frac{v_{jn}}{n}$$

shows that

$$\int_0^t v_{100}(s)ds$$

is approximated by t. Thus,

$$v(t) = 1$$

and

$$\int_0^t v(s)ds = \int_0^t v^2(s)ds = t.$$

Hence,

$$P(L_{100} \leq l)$$

is approximated by

$$\Phi\left(\frac{l-\lambda}{\sqrt{\lambda}}\right).$$

Let $l = k\lambda$. Then,

$$P(L_{100} \leq l) = \Phi((k-1)\lambda).$$

Assume that $\lambda = 0.1$. The following figure gives the plot of

$$P(L_{100} > 0.1(k-1))$$

for various values of $k \in (2,5)$. It is seen that as k gets large, then

$$P(L_{100} > l)$$

reduces to zero, linearly.

Insert Figure 1, here (See Appendix A)

Case 2. Here, v_{jn}'s are distributed uniformly on (0,1). The following figure gives the histogram of L_{20} based on two expressions

$$\sum_{j=1}^{20} v_{j,20} I_j \quad \text{(A)}$$

$$\sum_{i=1}^{20} \frac{\sum_{j=1}^{i} E_j}{\sum_{j=1}^{21} E_j} I_i \quad \text{(B)}$$

which are equivalent in distribution. Here, E_j's are independent and exponentially distributed with parameter 1.

Insert Figure 2, here (See Appendix A)

Case 3. Let $v_{j,100}$'s $j = 1, \dots, n = 100$ are independent and distributed as beta (2,3) law. Here, a Weibull distribution is fitted to $v_{1:100}$. Let a and b be the shape and scale parameters of Weibull distribution. The standard maximum likelihood method estimations of parameters contain optimizing a non-linear function, numerically. Here, a simple method is chosen based on quantile function which is given by

$$v_p = b(-\log(1-p))^{1/a}.$$

For two p_0, p_i and two quantiles v_{p_i} and v_{p_0}, notice that

$$\frac{v_{p_i}}{v_{p_0}} = \left(\frac{\log(1-p_i)}{\log(1-p_0)}\right)^{1/a}.$$

Let

$$z_i = \log\left(\frac{v_{p_i}}{v_{p_0}}\right)$$

and

$$y_i = \log\left(\frac{\log(1-p_i)}{\log(1-p_0)}\right).$$

Thus,

$$a = \frac{y_i}{z_i}.$$

For $i = 1, \dots, n$ quantiles, then the least square estimate of a is

$$\frac{\bar{y}_n}{\bar{z}_n}.$$

The mean of Weibull distribution is

$$b\Gamma\left(1 + \frac{1}{a}\right).$$

Thus, the estimate of b is given by

$$\frac{\bar{v}_n}{\Gamma\left(1+\frac{1}{a}\right)}.$$

It is seen that $a = 2.27, b = 0.04$.

Case 4. Suppose that distribution of p_{n-1} is

$$\text{beta}(a_{n-1}, a_{n-1}),$$

for some sequences a_n's. For $n = 25$, assume that $\frac{I_{25}}{25} = 0.3$. Then, the predictive expectation

$$E(I_{26}|I_k, k = 1, \dots, 25)$$

is given by

$$E_1 = 0.5 - 0.2\zeta_{25},$$

where

$$\zeta_{25} = \frac{25}{25 + 2a_{25}}.$$

The following plot gives the sensitivity analysis of mentioned conditional expectation with respect to a_{25}. Next, assume that n tends to infinity, and again let $\frac{I_n}{n} = 0.3$ and a_n/n tends to a. Therefore, ζ_n approaches to $\frac{1}{1+2a}$.

The following figure shows the plot of

$$E_2 == E(I_n|I_k, k = 1, \dots, n-1) =$$

$$= 0.5 - \frac{0.2}{1 + 2a}.$$

Insert Figure 3, here (See Appendix A)

Case 5. To approximate the correlation matrix of γ by correlation matrix of AR(1) process, let

$$\theta = \frac{\sum_{j=1}^{M} j\beta_j}{\sum_{j=1}^{M} j^2}.$$

For $M = 2$, when β_j's are independent and have common gamma distribution with parameters 1,1. The histogram of θ is given as follows. The exact moment generating function of θ is

$$\frac{25}{(5-t)(5-2t)}.$$

Using the Monte Carlo simulation, an approximate gamma distribution $gam(1.67, 0.375)$ is fitted to θ.

Insert Figure 4, here (See Appendix A)

Case 6. Let $v(t) = t$. Then, $L_n|\lambda$ has normal distribution with mean $\lambda/2$ and variance $\lambda/3$. Let $M = 2$, $\beta_j = 2, j = 1,2$ and $\beta = 1$. Therefore, λ has prior gamma distribution with parameters 1,1. The following plot gives the posterior density histogram of λ, using the Smith and Gelfand algorithm.

Insert Figure 5, here (See Appendix A)

Appendix A

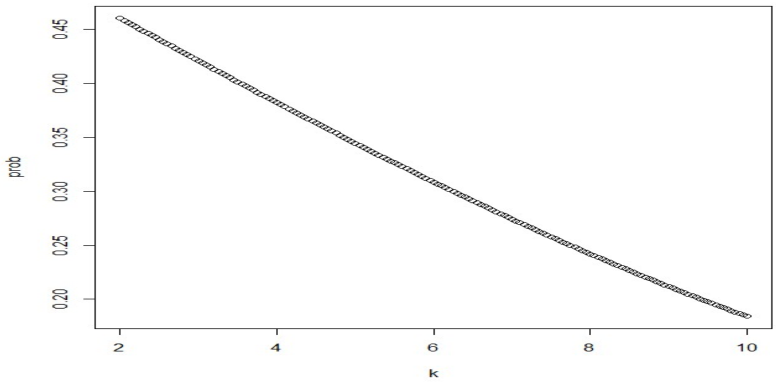
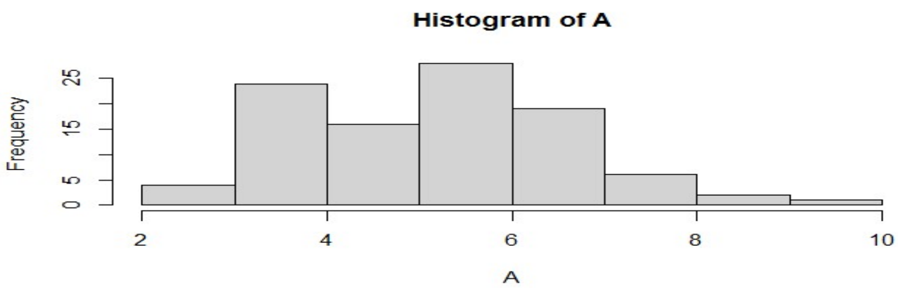


Figure 1. $P(L_{100} > 0.1(k-1))$.



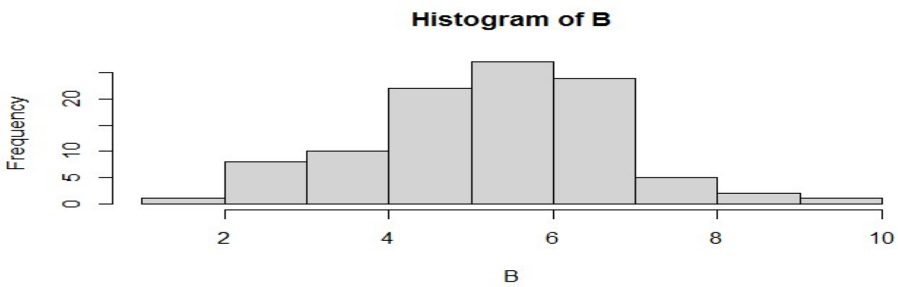


Figure 2. Histograms of L_n based on method A, B.

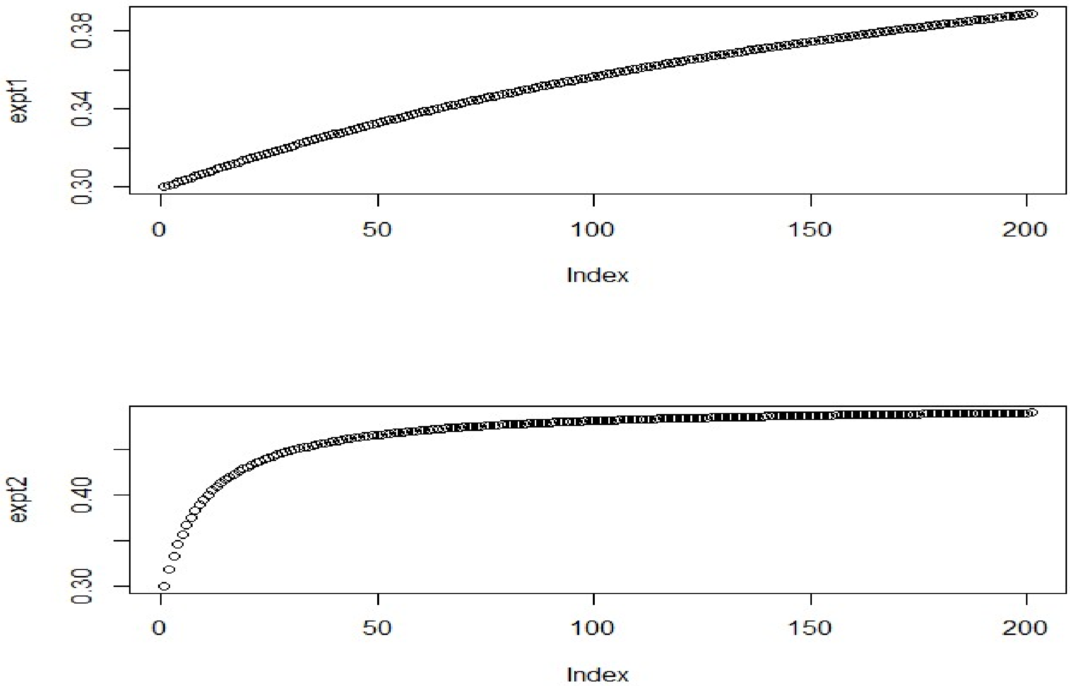


Figure 3. Plots of $E_i; i = 1,2$.

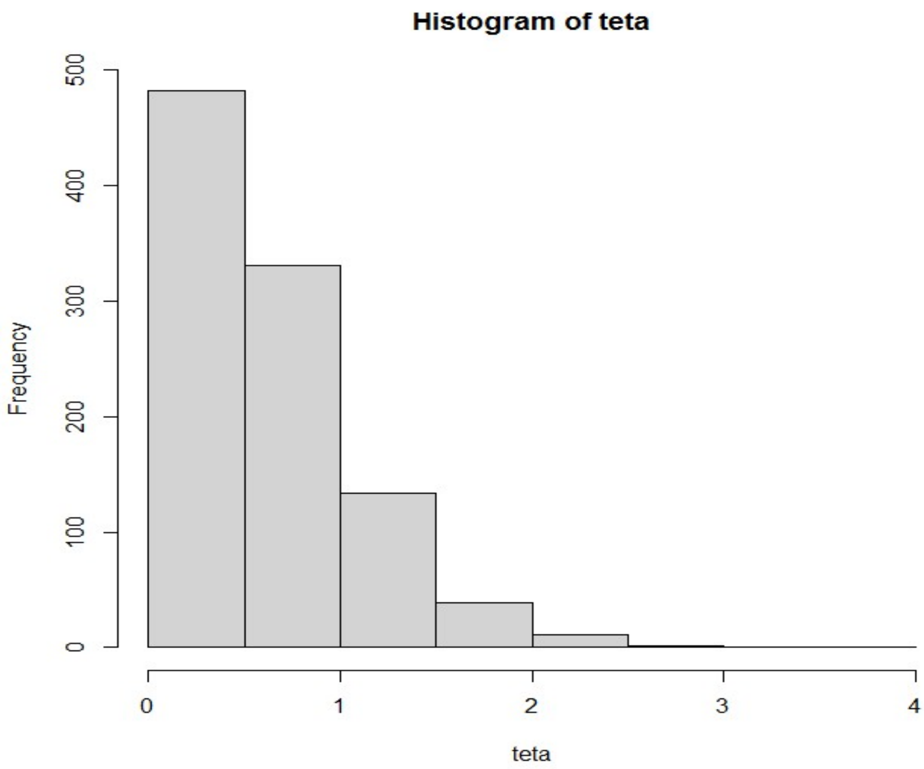


Figure 4. Histogram of θ.

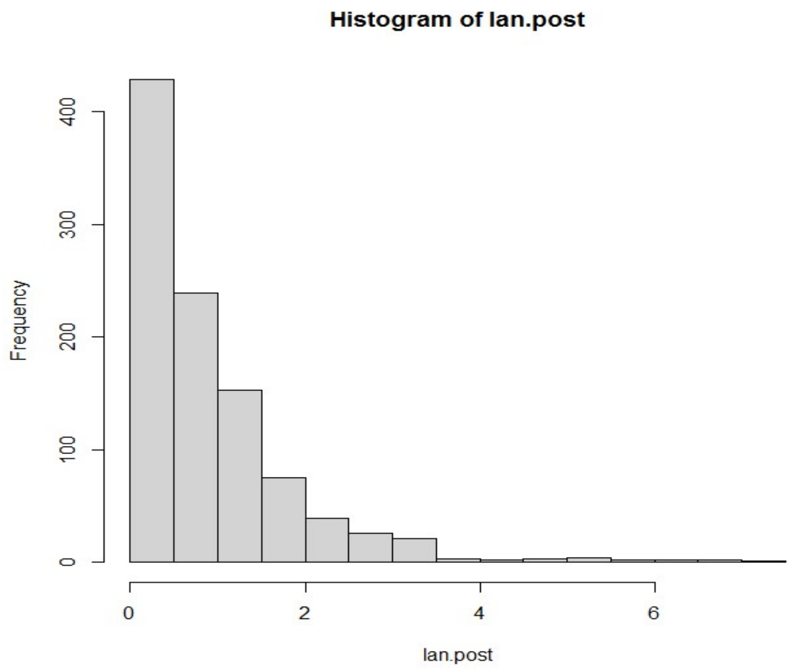


Figure 5. Histogram of Posterior of λ.

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